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An expanded Phillips theory and its application to differential operators☆

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Abstract

A new characterization of maximal accretive extensions of accretive operators is presented. It is based on expanded spaces of the abstract boundary spaces developed by R.S. Phillips, and is therefore called the expanded Phillips theory. In particular, when the operator is symmetric, a concrete realization of the expanded spaces is given. In applications of the expanded Phillips theory to ordinary differential equations, we characterize all maximal accretive extensions of singular symmetric ordinary differential operators of order $2n$.

Keywords: The expanded Phillips theory; Expanded space of the abstract boundary space; Maximal accretive extension; Friedrichs extension

1. Introduction

A linear densely defined operator T with domain $D(T)$ in a Hilbert space $\mathcal{H}$ is said to be *accretive* if

$$\lambda_0(T) := \inf\{\operatorname{Re}(Ty, y), \quad y \in D(T), ||y|| = 1\} \geqslant 0, \tag{1.1}$$

and *maximal accretive* if it is accretive and has no proper accretive extension. Hereafter, the constant $\lambda_0(T)$ will be called the lower bound of T. Note that T is called dissipative

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if $-T$ is accretive in the above sense (cf. [14, p. 279]). Furthermore, an operator T is said to be self-adjoint (symmetric) if $T = T^*$ $(T \subset T^*)$, where T^* is the adjoint of the operator T.

A fundamental problem concerning accretive operators is the *maximal accretive extension* problem: given an accretive operator T_0 in $\mathcal{H}$ with $\lambda_0(T_0) > 0$, one wishes to describe all maximal accretive operators T in $\mathcal{H}$ with $T_0 \subset T$. (Note that in this case all adjoints, T^*, of T are maximal accretive and satisfy $T^* \subset T_0^*$ (see [6, p. 119]), and are therefore called the *maximal accretive restrictions* of T_0^*.) The abstract theory of accretive (and dissipative) operators was developed originally by Phillips in a series of papers [5,22–25] and culminating in [26]. Phillips gave a complete and elegant solution to this problem in terms of certain abstract boundary spaces. The basic idea behind the Phillips theory involves identifying the graph of T_0 with a subspace P of $H := \mathcal{H} \times \mathcal{H}$ that is positive when H is equipped with a suitably chosen indefinite inner product. Let P' denote the orthogonal complement of P in H, and $\check{H}$ the completing space of P' with respect to the indefinite metric on H. (The space $\check{H}$ is called the *abstract boundary space*.) If the operator T is a maximal accretive extension, then the theory shows that the graph of T^* is associated with a maximal negative subspace of $\check{H}$.

In applications of the Phillips theory to differential equations, the extension (or restriction) problem now reduces in each case to finding a suitable explicit realization of the abstract boundary space $\check{H}$. However, relatively little use appears to have been made of this result, partly because in practice it is not always easy to produce concrete realizations of $\check{H}$. Evans and Knowles [7,8] solved the maximal accretive extension problem for regular and singular limit-point differential operators by making use of the theory, under the assumption that (roughly speaking) functions in the appropriate maximal domains all have finite energy integrals (see [5,7,8]). It should be noted that the results of Evans and Knowles are not suitable for the singular differential operators with middle and maximal deficiency indices, because, in these cases, all functions need not have finite energy integrals (see [3,15]). Furthermore, the assumption of "finite energy integrals" is stronger than the basic condition $\lambda_0(T_0) > 0$. In particular, Evans and Knowles [8, p. 265] raised the following problem: It is not known what happens if the assumption of "finite energy integrals" is dropped. Thus, the extension problem is open even for the simplest singular differential operators, $ly = -y'' + q(t)y$, of limit-circle type.

To overcome the above difficulties and with a view to applications to ordinary differential equations, the purpose of this paper is to generalize and revise the Phillips theory. For an accretive operator T_0 with $\lambda_0(T_0) > 0$ and finite deficiency index m, i.e.,

$$\operatorname{def}(T_0 - \lambda I) := \dim(\ker(T_0^* - \lambda I)) =: m < \infty, \tag{1.2}$$

where $\operatorname{Re}(\lambda) < \lambda_0(T_0)$ (see [6, p. 100]), we introduce the expanded space X of the abstract boundary space $\check{H}$, which is an extension space of $\check{H}$ and preserves the positive index, and prove that we can use the maximal negative subspaces of X instead of those of $\check{H}$ to characterize the maximal accretive extensions of T_0. This expanded

Phillips theory is of particular importance in applications and theories: (i) It avoids the complex process of realizing the abstract boundary space $\check{H}$, (ii) and enhances the flexibility of the original Phillips theory in applications. As an example, when T_0 is a symmetric operator, we concretely construct an expanded space X in terms of the Friedrichs extension [11] of T_0, a direct sum decomposition of the domain of the adjoint T_0^* of T_0 (see Lemma 2.5 below) and the boundary mapping of T_0^* (see [28,31]; and also Definition 2.6 below). Here, we indicate that the idea of this concrete realization originates from [28,31,34] which respectively concern the maximal accretive realizations of regular Sturm–Liouville differential operators by an expanded space technique, and the self-adjoint extensions of symmetric operators by the boundary mapping. Note that this concrete realization of X is associated directly with the lower bound $\lambda_0(L_0)$ of T_0 and therefore, the assumption of "finite energy integrals" may be dropped in applications to differential operators.

The expanded Phillips theory can be conveniently applied to ordinary differential operators. Let l be a formally symmetric differential expression and $L_{\min}$ denote the minimal operator associated with l in the corresponding L^2 space. By the way, once all maximal accretive restrictions, L, of $L_{\min}^*$ are known, the explicit form of all maximal accretive extensions of $L_{\min}$ can be easily produced, say, via using the adjoint construction technique of Brown and Krall [4,8,Appendix]. Thus, in the present paper, we will mainly be concerned with the maximal accretive restriction problem for differential operators. Making use of the Friedrichs extension L_F of $L_{\min}$ we can completely describe all maximal accretive restrictions of $L_{\min}^*$. Note that, in applications of the expanded Phillips theory to symmetric differential operators $L_{\min}$, a major problem is to characterize the boundary conditions of the Friedrichs extension L_F. The Friedrichs extension has been studied by a great many of authors in the context of various differential operators. See, for example, [1,12,13,16–21,35]. Thus, in conjunction with these results, we can solve the maximal accretive restriction problem for the associated differential operators. For example, in this paper, by using of the work of Marletta and Zettl [17], which applied the principal solutions of $2n$th order singular differential equations to describe the Friedrichs extension, we characterize all maximal accretive extensions of $L_{\min}$. We note that, in this case, the deficiency indices of $L_{\min}$ do not need to be of limit-point type. Particularly, when $n = 1$, we give a satisfactory solution to the open problem of Evans and Knowles.

The organization of the paper is as follows. In Section 2 we present an outline of the Phillips theory and the expanded Phillips theory. In particular, Theorem 2.9 gives a concrete realization of the expanded space when T_0 is a symmetric accretive operator. In Section 3, we characterize the maximal accretive extensions of singular symmetric differential operators in Theorem 3.11 and Corollaries 3.13, 3.16.

2. The expanded Phillips theory

Throughout this section, let T_0 denote a closed densely defined accretive operator in the Hilbert space $\mathcal{H}$ with the lower bound $\lambda_0(T_0)$. We assume that $\lambda_0(T_0) > 0$ and $\operatorname{def}(T_0) = m < \infty$, defined by (1.1) and (1.2), respectively. In this section we first summarize the Phillips extension theory and define the expanded space X of the abstract boundary space $\check{H}$. Then we give a new characterization for the maximal accretive extensions of T_0, which we call the expanded Phillips theory. In particular, when T_0 is symmetric, we obtain a concrete realization of the expanded space X.

Let $\mathbf{R}$ be the real line, $\mathbf{C}$ be the complex field and $\mathbf{C}^m = \{\alpha = (c_1, \ldots, c_m) : c_i \in \mathbf{C}, i = 1, \ldots, m\}$. We write a matrix A with m rows and n columns as $A = (a_{ij})_{m\times n}$ or $A = (a_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$, where a_{ij} is the element of A appearing in the ith row and jth column. In the case when $m = n$, we simply write $A = (a_{ij})_n$ or $A = (a_{ij})_{1\leqslant i,j\leqslant n}$. If all elements of A are zeros, we write A as $0_{m\times n}$. Let A^T and A^* denote the transpose and Hermitian adjoint of A, respectively.

2.1. The Phillips theory

We outline the Phillips extension theory for maximal accretive operators. The treatment here follows that given in [26] except for the trivial change from dissipative to accretive operators.

Set $H = \mathcal{H} \times \mathcal{H}$ and for $\check{u} = \{u_1, u_2\}$ and $\check{v} = \{v_1, v_2\}$ in H define the indefinite inner product (called the Q-inner product also)

$$Q(\check{u}, \check{v}) = (u_2, v_1) + (u_1, v_2). \tag{2.1}$$

With respect to this Q-inner product, H has the following fundamental decomposition (see [2, p. 24]):

$$H = H_+ \oplus H_-, \tag{2.2}$$

where

$$H_+ = \{\{u, u\}, u \in \mathcal{H}\} \quad \text{and} \quad H_- = \{\{v, -v\}, v \in \mathcal{H}\},$$

and each $\check{u} = \{u_1, u_2\}$ in H may be written as $\check{u} = \check{u}_+ + \check{u}_-$, where

$$\check{u}_+ = \left\{\frac{1}{2}(u_1 + u_2), \frac{1}{2}(u_1 + u_2)\right\} \quad \text{and} \quad \check{u}_- = \left\{\frac{1}{2}(u_1 - u_2), \frac{1}{2}(u_2 - u_1)\right\}.$$

Also H_+ and H_- are orthogonal and Q-orthogonal subspaces. It is not hard to see that H is a Krein space (see [2, p. 100]).

Note that a linear manifold K of H is called positive (negative) if

$$Q(\check{u}, \check{u}) \geqslant 0 \quad (\leqslant 0) \qquad \text{for all } \check{u} \in K.$$

A subspace of H will mean always a closed linear manifold with respect to the induced Hilbert norm $(\check{u}, \check{u}) := Q(\check{u}_+, \check{u}_+) - Q(\check{u}_-, \check{u}_-)$ on H. A subspace K of H is maximal positive if it is positive and is not properly contained in another positive subspace. Furthermore, if K is a maximal positive subspace of H, then the Q-orthogonal complement $K^{\perp} := \{\check{u} \in H : Q(\check{u}, \check{v}) = 0, \text{ for all } \check{v} \in K\}$ is a maximal negative subspace. For proofs of these and other well-known facts the reader is referred to [2].

Let $P = G(T_0)$, the graph of T_0 in $\mathcal{H}$. As $\lambda_0(T_0) > 0$, it follows that P is a positive definite linear manifold of H. Let P' denote the Q-orthogonal complement of P in H. By [26, Theorem 3.1] it follows that $P' = G(-T_0^*)$ can be decomposed into orthogonal and Q-orthogonal strictly positive and strictly negative parts: $P' = M_+ \oplus M_-$, where

$$M_+ = P' \cap H_+ = \{\{y, -T_0^* y\} : T_0^* y = -y, \ y \in D(T_0^*)\} \tag{2.3}$$

is intrinsically complete, i.e., complete with respect to the Q-norm (cf. [2, p. 71]). The negative subspace, M_-, however, need not be intrinsically complete. Let the intrinsic completion of M_- be denoted by $\bar{M}_-$. Then we can define the abstract boundary space, $\check{H}$, by

$$\check{H} = M_+ \oplus \bar{M}_-. \tag{2.4}$$

Clearly, from (2.3) and $\operatorname{def}(T_0 + I) = m < \infty$, $\check{H}$ is a Pontryagin space of positive index m (i.e., a Π_m space, see [2, p. 184]).

With the above definitions and notations, Phillips proved the following basic result:

Lemma 2.1. *Let $\lambda_0(T_0) > 0$. An operator T is a maximal accretive extension of T_0 if and only if there exists a maximal negative subspace $\check{N}$ of $\check{H}$ such that*

$$G(-T^*) = \check{N} \cap P'. \tag{2.5}$$

Proof. This is given in [26, Theorem 5.2]. □

This lemma is known as the *Phillips theory*. The theory shows that each possible accretive maximal extension of T_0 (equivalently, the graph of its adjoint) can be identified with a certain explicitly given (via the abstract boundary space $\check{H}$) maximal negative space of P'. In applications of the abstract theory, most of the difficulties are encountered in constructing the explicit realization of the abstract boundary space and, in practice, it is not always easy to produce the explicit realization (cf. [7,8]). In the present paper such difficulties are avoided by using *an expanded space trick*. Particularly, we will construct an inner product-preserving expanded space, X, of $\check{H}$ such that both X and $\check{H}$ have the same positive index, and therefore, finding the maximal negative subspaces of $\check{H}$ is equivalent to doing so for X. The construction of X effectively eliminates the difficulties of realizing $\check{H}$ concretely. We will clarify this in the next subsection.

2.2. The expanded Phillips theory

Definition 2.2. Let $X := (X, Q_1(\cdot,\cdot))$ be a Krein space. Then X is called an *expanded space* of the abstract boundary space $\check{H}$ if X satisfies
(i) X contains $\check{H}$ and extends its Krein space structure, that is, $\check{H} \subseteq X$ and, for any $\check{y}, \check{z} \in \check{H}$, we have

$$Q(\check{y}, \check{z}) = Q_1(\check{y}, \check{z}), \tag{2.6}$$

(ii) both X and $\check{H}$ have the same positive index, that is, X is a Π_m space.

Remark 1. Although the definition of the expanded space X is based on the abstract boundary space $\check{H}$, in fact, because the space $\check{H}$ is derived from completing the graph $P' = G(-T_0^*)$ with respect to the Q-norm, in practice we may directly construct the expanded space X in terms of P', without bothering to realize $\check{H}$.

Remark 2. In order to distinguish the elements of X and $\check{H}$, we shall always denote the elements in X by $\hat{u}$. Obviously, we identify $\check{H}$ with a subspace of X via the association

$$\check{u} \leftrightarrow \hat{u}, \tag{2.7}$$

where $\check{u} \in \check{H}$ and $\hat{u} \in X$. Thus, as $\check{u} \in \check{H}$, we may write it as $\hat{u} \in X$.

Now we state and prove one of the main results of this section.

Theorem 2.3. *Let $\lambda_0(T_0) > 0$, $\operatorname{def}(T_0) = m < \infty$ and X be an expanded space of $\check{H}$. Then an operator T is a maximal accretive extension of T_0 if and only if there exists a maximal negative subspace $\hat{N}$ of X such that*

$$G(-T^*) = \hat{N} \cap P'. \tag{2.8}$$

Proof. *Necessity*: Let T be a maximal accretive extension of T_0. By Lemma 2.1 there exists a maximal negative subspace $\check{N}$ of $\check{H}$ such that (2.5) holds. Note that by Definition 2.2 the spaces $\check{H}$ and X all are Π_m space and satisfy $\check{H} \subseteq X$. If X_0 in $\check{H}$ is the Q-orthogonal (also, Q_1-orthogonal) complement of $\check{N}$, then X_0 is a maximal positive subspace of $\check{H}$ with index m and also of X. Let $\hat{N}$ denote the Q_1-orthogonal complement of X_0 in X. Then from (2.6) we have that $\check{N} \subseteq \hat{N}$ and $\hat{N}$ is a maximal negative subspace of X. We see that

$$\hat{N} \cap P' = \hat{N} \cap G(-T_0^*) =: G(-\hat{T}^*)$$

is the graph of some operator $-\hat{T}^*$ satisfying $\hat{T}^* \subseteq T_0^*$. As a consequence, $\hat{T}^* \supseteq T^*$ is an accretive operator. Since T is a maximal accretive extension, from [6, p. 120] the adjoint T^* is a maximal accretive restriction of T_0^*. Thus, from the definition of a maximal accretive operator, it follows that $\hat{T}^* = T^*$ and therefore $\hat{T} = T$. This completes the proof of necessity.

Sufficiency: Let $\hat{N}$ be a maximal negative subspace of X. Then from (2.1), (2.8) and Definition 2.2, for any $u \in D(T^*)$, we have

$$2\mathrm{Re}(T^* u, u) = (T_0^* u, u) + (u, T_0^* u) = -Q(\check{u}, \check{u}) = -Q_1(\hat{u}, \hat{u}) \geqslant 0,$$

where $\check{u} = \{u, -T_0^* u\}$. Therefore, T^* is an accretive operator. In the following we further verify that T^* is maximal accretive. From [6, Theorem 6.5] we only need to show that range $(T^* + I) = \mathcal{H}$, that is, for any given $f \in \mathcal{H}$, we prove the existence of a solution u in $D(T^*)$ to the equation $(T^* + I)u = f$.

From $\mathrm{def}(T_0 + I) = m$, we choose w_i, $1 \leqslant i \leqslant m$, in $D(T_0^*)$ to be linearly independent solutions of $(T_0^* + I)u = 0$ and u_0 in $D(T_0^*)$ to be a particular solution of $(T_0^* + I)u = f$, so, the general solution of $(T_0^* + I)u = f$ is given by $u := u_0 + \sum_{i=1}^m c_i w_i$, $c_i \in \mathbf{C}$. Furthermore, since X is a Π_m space and $\hat{N}$ is a maximal negative subspace of X, there exists a maximal positive subspace X_0 of X, which is generated by a set $\{\hat{\varphi}_1, \ldots, \hat{\varphi}_m\}$, such that $\hat{N} = X_0^{\perp}$ (Q_1-orthogonal complement of X_0 in X). Let

$$Q_1(\hat{u}, \hat{\varphi}_j) = 0, \qquad 1 \leqslant j \leqslant m, \tag{2.9}$$

where $\hat{u} = \check{u} = \{u, -T_0^* u\} \in P'$. Thus

$$(c_1, \ldots, c_m)[(Q_1(\hat{w}_i, \hat{\varphi}_j))_{1 \leqslant i,j \leqslant m}] = -(Q_1(\hat{u}_0, \hat{\varphi}_1), \ldots, Q_1(\hat{u}_0, \hat{\varphi}_m)). \tag{2.10}$$

We claim that the rank of the matrix $(Q_1(\hat{w}_i, \hat{\varphi}_j))_{1 \leqslant i,j \leqslant m}$ is m. If not, then there is a nontrivial element $w = \sum_{i=1}^m a_i w_i$ satisfying $Q_1(\hat{w}, \hat{\varphi}_j) = 0$, $1 \leqslant j \leqslant m$. This implies that $\hat{w} \in X_0^{\perp}$ and therefore $Q_1(\hat{w}, \hat{w}) \leqslant 0$. On the other hand, since $(T_0^* + I)w = 0$ and $w \neq 0$, from (2.6) and (2.1) we obtain

$$Q_1(\hat{w}, \hat{w}) = Q(\check{w}, \check{w}) = 2(w, w) > 0.$$

This contradicts the inequality $Q_1(\hat{w}, \hat{w}) \leqslant 0$. Thus from (2.9) we obtain that the equation $(T_0^* + I)u = f$ has a unique solution in $D(T^*)$ and range $(T^* + I) = \mathcal{H}$. As a consequence, T^* is a maximal accretive restriction of T_0^*. By [6, Theorem 6.6] it follows that T is a maximal accretive extension of T_0, thus completing the proof of Theorem 2.3. $\square$

Remark. Theorem 2.3 shows that one can describe the maximal accretive extensions of T_0 by finding the maximal negative subspaces of X instead of those of $\check{H}$. Because of this, Theorem 2.3 is called the *expanded Phillips theory*. In the following we will see that replacing $\check{H}$ with the expanded space X is of great importance in applications, especially in the matter of finding concrete realizations. The added convenience and flexibility expedite applications of the theory to differential equations.

2.3. A concrete realization of X

In what follows, when T_0 is a symmetric accretive operator, we will present a concrete realization of X to show that the expanded Phillips theory can be applied effectively. This realization is based on the Friedrichs extension of T_0, a direct sum decomposition of the domain of the adjoint of T_0 and the boundary mapping of T_0^*. Next, we give a detailed discussion.

If T_0 is a closed symmetric operator with $\lambda_0(T_0) > -\infty$, then $T_0 \subset T_0^*$ and the deficiency indices of T_0 are equal, which can written $m_+ = m_- =: m$ (say, $\operatorname{def}(T_0) = m$), where

$$m_\pm := \dim(\ker(T_0^* \mp iI)).$$

Definition 2.4. Let T_0 be a symmetric operator which is bounded below. The operator T_F is called the *Friedrichs extension* of T_0 if its domain $D(T_F)$ consists of all y in $D(T_0^*)$ such that there exists a sequence y_k in $D(T_0)$ satisfying
(a) $y_k \to y$ in $\mathcal{H}$ as $k \to \infty$,
(b) $(T_0(y_k - y_n), y_k - y_n) \to 0$ as $k, n \to \infty$,
and the operator T_F is the restriction of T_0^* to $D(T_F)$.

It is well known [cf. 32, p. 120] that if the symmetric operator T_0 is bounded below then its Friedrichs extension always exists, and it is a self-adjoint extension of T_0, which preserves the lower bound of T_0, i.e.,

$$\lambda_0(T_F) = \lambda_0(T_0). \tag{2.11}$$

Lemma 2.5. *Let $\lambda_0(T_0) > 0$ and T_F be the Friedrichs extension of T_0. Then*

$$D(T_0^*) = D(T_F) \dot{+} \ker(T_0^*), \tag{2.12}$$

where the symbol $\dot{+}$ indicates the sum is direct.

Proof. Let $N_\pm(i) = \{u \in \mathcal{H} : (T_0^* \mp iI)u = 0\}$. From the first formula of von Neumann (see [32, p. 237]), we have

$$D(T_0^*) = D(T_0) \dot{+} N_+(i) \dot{+} N_-(i). \tag{2.13}$$

Since T_F is a self-adjoint extension of T_0 and $\operatorname{def}(T_0) = m$, T_F is a m dimensional extension of T_0 and there exist $\varphi_i \in N_+(i)\dot{+}N_-(i)$, $1 \leqslant i \leqslant 2m$, such that

$$D(T_F) = D(T_0)\dot{+}\operatorname{span}\{\varphi_{m+1}, \ldots, \varphi_{2m}\}, \tag{2.14}$$

$$D(T_0^*) = D(T_F)\dot{+}\operatorname{span}\{\varphi_1, \ldots, \varphi_m\}. \tag{2.15}$$

Here, $\varphi_1, \ldots, \varphi_m$ are linearly independent relative to $D(T_F)$ (cf. [32, p. 238]). Note that by [32, p. 115] the condition $\lambda_0(T_0) > 0$ yields $\dim(\ker(T_0^*)) = m$, and there exist m linearly independent elements θ_i in $D(T_0^*)$ such that $\operatorname{span}\{\theta_1, \ldots, \theta_m\} = \ker(T_0^*)$. By (2.15) each θ_i has a unique representation

$$\theta_i = u_{Fi} + \sum_{j=1}^{m} c_{ij}\,\varphi_j, \qquad u_{Fi} \in D(T_F). \tag{2.16}$$

Let $C = (c_{ij})_{1 \leqslant i,j \leqslant m}$. We show that rank $C = m$. If it is not true, then there are d_i in $\mathbf{C}$, $1 \leqslant i \leqslant m$, which are not all zero, satisfying $\sum_{i=1}^{m}\sum_{j=1}^{m} d_j c_{ij} \varphi_j = 0$. This implies

$$0 \neq \theta_0 := \sum_{i=1}^{m} d_i \theta_i = \sum_{i=1}^{m} d_i u_{Fi} \in D(T_F). \tag{2.17}$$

Furthermore, since T_F is a bound-preserving self-adjoint extension of T_0, then $0 \in \rho(T_F)$ (resolvent set) and $\ker(T_F) = 0$. This contradicts (2.17), and therefore rank $C = m$. Thus, we may solve φ_j from (2.16), and each φ_j has a unique representation

$$\varphi_j = y_{Fj} + \sum_{s=1}^{m} b_{js}\theta_s, \qquad y_{Fj} \in D(T_F), \quad 1 \leqslant j \leqslant m. \tag{2.18}$$

By (2.15) and (2.18), it follows that, for each $y \in D(T_0^*)$,

$$\begin{aligned} y &= y'_F + \sum_{j=1}^{m} a'_j \varphi_j \\ &= y'_F + \sum_{j=1}^{m} a'_j \left(y_{Fj} + \sum_{s=1}^{m} b_{js}\theta_s \right) \\ &= y_F + \sum_{s=1}^{m} a_s \theta_s, \end{aligned}$$

where y'_F, $y_F := y'_F + \sum_{j=1}^{m} a'_j y_{Fj} \in D(T_F)$ and $a_s = \sum_{j=1}^{m} a'_j b_{js}$. By the uniqueness of the representations of y and φ_j (see (2.15) and (2.18)), Lemma 2.5 is proved. $\square$

Since T_F is a bound-preserving self-adjoint extension of T_0, if $\lambda_0(T_0) > 0$, then the symmetric sesquilinear form $(T_F\cdot,\cdot)$ is a positive definite inner product on the linear manifold $D(T_F)$. Denote its completing space by

$$H_F := (H_F, (\cdot,\cdot)_D). \tag{2.19}$$

By the way, from Definition 2.4, we see that the linear manifold $D(T_0)$ is dense in H_F with respect to the inner product $(\cdot,\cdot)_D$. Furthermore, by Lemma 2.5, each $u \in D(T_0^*)$ can be uniquely represented as

$$u = u_F + \sum_{i=1}^{m} c_i\theta_i, \qquad y_F \in D(T_F), \quad \operatorname{span}\{\theta_1,\dots,\theta_m\} = \ker(T_0^*). \tag{2.20}$$

Because of this, we will denote the above inner product henceforth by $(u,v)_D^F$ for any $u, v \in D(T_0^*)$, that is,

$$(u,v)_D^F = (T_F u_F, v_F), \qquad u, v \in D(T_0^*). \tag{2.21}$$

Definition 2.6. Let T_0 be a symmetric operator with $\operatorname{def}(T_0) = m < \infty$. If the linear mapping $\Gamma(\cdot) : D(T_0^*) \mapsto \mathbf{C}^{2m}$ is surjective and all u_0 in $D(T_0)$ satisfy $\Gamma(u_0) = 0$, then $\Gamma(u)$ is called the *boundary vector* of u in $D(T_0^*)$ and Γ the *boundary mapping* of T_0^*.

In analogy with ordinary differential operators [cf. 18], let k be an integer with $0 \leqslant k \leqslant 2m$ and M be a $k \times 2m$ matrix over $\mathbf{C}$ with rank $M = k$. For any such M and a boundary mapping $\Gamma(\cdot)$ we define an operator $T(M)$ from H into itself by

$$\begin{aligned} D(T(M)) &= \{u \in D(T_0^*) : M\Gamma^*(u) = 0\}, \\ T(M)u &= T_0^* u \qquad (u \in D(T(M))). \end{aligned} \tag{2.22}$$

When $k = 0$ we have $M = 0$ and $T(M) = T_0^*$, and when $k = 2m$ we have $T(M) = T_0$. Here the matrix M and $M\Gamma^*(u) = 0$ may be called the boundary matrix and a boundary condition respectively. From (2.22) it is clear that $D(T(M))$ is a linear submanifold of $D(T_0^*)$ and satisfies $T_0 \subseteq T(M) \subseteq T_0^*$ for any boundary matrix M. It was proved [31, Lemma 4] that if we choose appropriate boundary matrices then all self-adjoint extensions of T_0 can be described in terms of (2.22).

Proposition 2.7. *Let T_0 be a symmetric operator with* $\operatorname{def}(T_0) = m < \infty$. *If both Γ_1 and Γ_2 are boundary mapping of T_0^*, then there exists a $2m \times 2m$ nonsingular matrix Δ such that*

$$\Gamma_1(u) = \Gamma_2(u)\Delta, \qquad for\ all\ u \in D(T_0^*).$$

Proof. See [28, Lemma 2]. □

Proposition 2.8. *Under the assumptions that T_0 is a symmetric operator with $\lambda_0(T_0)>0$ and $\mathrm{def}(T_0)=m<\infty$, and Γ is a boundary mapping of T_0^*, there exist two $2m\times 2m$ Hermitian matrices A and B such that for any $y\in D(T_0^*)$ the following identities hold*

$$2\mathrm{Im}(T_0^*u,u)=\Gamma(u)A\Gamma^*(u), \tag{2.23}$$

$$2\mathrm{Re}(T_0^*u,u)=2(u,u)_D^F+\Gamma(u)B\Gamma^*(u), \tag{2.24}$$

where both matrices A and B are nonsingular and have zero signature.

Proof. Since $\mathrm{def}(T_0)=m$ and $\lambda_0(T_0)>0$, we have $\dim(\ker(T_0^*))=m$ and $R(T_0)\perp\ker(T_0^*)$. Let $\ker(T_0^*)=:\mathrm{span}\{\theta_1,\ldots,\theta_m\}$. For any $u\in D(T_0^*)$, it follows from (2.20) and (2.14) that

$$u=u_F+\sum_{i=1}^{m}c_i\theta_i \quad\text{and}\quad u_F=u_0+\sum_{i=m+1}^{2m}c_i\varphi_i, \tag{2.25}$$

where $u_F\in D(T_F)$, $u_0\in D(T_0)$ and $\varphi_i\in N_+(i)\dot{+}N_-(i)$. For convenience, we write φ_{m+i} as θ_{m+i}, $1\leqslant i\leqslant m$. By (2.25) and (2.21) we have $u=u_0+\sum_{i=1}^{2m}c_i\theta_i$ and

$$\begin{aligned}(T_0^*u,u)&=\left(T_0^*(u_F+\sum_{i=1}^{m}c_i\theta_i),u_F+\sum_{j=1}^{m}c_j\theta_j\right)\\&=(T_0^*u_F,u_F)+\left(T_0^*(u_0+\sum_{i=m+1}^{2m}c_i\theta_i),\sum_{j=1}^{m}c_j\theta_j\right)\\&=(u,u)_D^F+\sum_{j=1}^{m}\bar{c}_j(T_0u_0,\theta_j)+\sum_{i=m+1}^{2m}\sum_{j=1}^{m}c_i\bar{c}_j(T_0^*\theta_i,\theta_j)\\&=(u,u)_D^F+\sum_{i=m+1}^{2m}\sum_{j=1}^{m}c_i\bar{c}_j(T_0^*\theta_i,\theta_j).\end{aligned} \tag{2.26}$$

Thus, if we write $\Gamma_1(u)=(c_1,\ldots,c_{2m})$, then (2.26) implies

$$2\mathrm{Im}(T_0^*u,u)=-i\Gamma_1(u)\hat{B}_0\Gamma_1^*(u), \tag{2.27}$$

$$2\mathrm{Re}(T_0^*u,u)=2(u,u)_D^F+\Gamma_1(u)B_0\Gamma_1^*(u), \tag{2.28}$$

where

$$\hat{B}_0 = \begin{pmatrix} 0 & -B_{00}^* \\ B_{00} & 0 \end{pmatrix} \quad \text{and} \quad B_0 = \begin{pmatrix} 0 & B_{00}^* \\ B_{00} & 0 \end{pmatrix} \tag{2.29}$$

with $B_{00} = ((T_0^* \theta_{m+i}, \theta_j))_{1 \leqslant i,j \leqslant m}$. Note that $\theta_1, \ldots, \theta_{2m}$ are linearly independent relative to $D(T_0)$. It is not hard to see from (2.25) and Definition 2.6 that Γ_1 is a boundary mapping of T_0^*. From [31, Lemma 4] we have rank $\hat{B}_0 = 2m$ and rank $B_{00} = m$. This therefore shows that rank $B_0 = 2m$ and its signature is 0. Furthermore, by Proposition 2.7, there exists a nonsingular $2m \times 2m$ matrix Δ such that $\Gamma_1(u) = \Gamma(u)\Delta$ for all u in $D(T_0^*)$. Substituting this into (2.27) and (2.28), we obtain $A = -i\Delta\hat{B}_0\Delta^*$, $B = \Delta B_0 \Delta^*$, and (2.23), (2.24). This completes the proof. □

If $\lambda_0(T_0) > 0$ and $\mathrm{def}(T_0) = m < \infty$, keeping in mind (2.19) and Proposition 2.8, we define the product space

$$X = H_F \times \mathbf{C}^{2m}, \tag{2.30}$$

and for any $\hat{y}_k = \{y_k, \alpha_k\} \in X$, $y_k \in H_F$, $\alpha_k \in \mathbf{C}^{2m}$, $k = 1, 2$, we define the indefinite inner product $Q_1(\cdot, \cdot)$ on X by

$$Q_1(\hat{y}_1, \hat{y}_2) = -2(y_1, y_2)_D - \alpha_1 B \alpha_2^*. \tag{2.31}$$

Here the matrix B satisfies (2.24), which associates the boundary mapping Γ to T_0^*.

We are now in a position to prove

Theorem 2.9. *If T_0 is a symmetric operator with $\lambda_0(T_0) > 0$ and* $\mathrm{def}(T_0) = m < \infty$, *and the space X is defined by* (2.30) *with inner product* (2.31), *then*
(i) *X is a Pontryagin space of positive index m,* (*i.e., a Π_m space*). *Therefore, both space X and $\check{H}$ have the same positive index*;
(ii) *X is an expanded space of $\check{H}$.*

Proof. (i) From Proposition 2.8, one can show that $B = G\left(\begin{smallmatrix} I_m & 0 \\ 0 & -I_m \end{smallmatrix}\right)G^*$, where I_m is the identity matrix of order m and G is a $2m$ square nonsingular matrix. Thus we have the decomposition

$$X = X_+ \oplus X_-, \tag{2.32}$$

where

$$X_+ = \{\{0, (0_{1\times m}, a_{m+1}, \ldots, a_{2m})G^{-1}\} : a_i \in \mathbf{C}, m+1 \leqslant i \leqslant 2m\},$$

$$X_- = \{\{u, (a_1, \ldots, a_m, 0_{1\times m})G^{-1}\} : a_i \in \mathbf{C}, 1 \leqslant i \leqslant m,\ u \in H_F\}.$$

Any $\hat{u} = \{u, \alpha\} \in X,\ u \in H_F,\ \alpha \in \mathbf{C}^{2m}$, can be written as $\hat{u} = \hat{u}_+ + \hat{u}_-$, where

$$\hat{u}_+ = \{0, (0_{1\times m}, \alpha_{10})G^{-1}\} \in X_+ \quad \text{and} \quad \hat{u}_- = \{u, (\alpha_{01}, 0_{1\times m})G^{-1}\} \in X_-$$

with $\alpha_{01}, \alpha_{10} \in \mathbf{C}^m$ and $(\alpha_{01}, \alpha_{10}) = \alpha G^{-1}$. The spaces X_+ and X_- are positive and negative definite, respectively, and are intrinsically complete since

$$Q_1(\hat{y}_+, \hat{y}_+) = \alpha_{10}\alpha_{10}^* \quad \text{and} \quad -Q_1(\hat{y}_-, \hat{y}_-) = 2(y, y)_D + \alpha_{01}\alpha_{01}^*. \tag{2.33}$$

This implies assertion (i).

(ii) Let $\check{u}_i = \{u_i, -T_0^* u_i\} \in P'$ and $\hat{u}_i = \{u_{Fi}, \Gamma(u_i)\} \in X,\ u_i \in D(T_0^*),\ i = 1, 2$, where the row vectors $\Gamma(u_i) \in \mathbf{C}^{2m}$ and the elements $u_{Fi} \in H_F$ are defined by (2.24) and (2.20) respectively. From (2.1) and (2.31) we have

$$\begin{aligned} Q(\check{u}_1, \check{u}_2) &= (-T_0^* u_1, u_2) + (u_1, -T_0^* u_2) \\ &= -2(u_1, u_2)_D^F - \Gamma(u_1) B \Gamma^*(u_2) \\ &= Q_1(\hat{u}_1, \hat{u}_2). \end{aligned} \tag{2.34}$$

Thus, we can identify P' with a subspace of X via the association

$$\check{u} \leftrightarrow \hat{u}, \tag{2.35}$$

where $\check{u} = \{u, -T_0^* u\} \in P',\ \hat{u} = \{u_F, \Gamma(u)\} \in X,\ u \in D(T_0^*)$. Thus the correspondence (2.35) is an one-to-one inner product-preserving map of P' into X. Since the space $\check{H}$ is obtained by completing the graph $P' = G(-T_0^*)$ with respect to the Q-norm (and, also Q_1-norm), so, we can write

$$P' \subset \check{H} \subset X. \tag{2.36}$$

Here $\check{H}$ can be regarded as a complete subspace of X. This combined with (i) completes the proof. $\square$

The above theorem presents a concrete realization of the expanded space X when T_0 is symmetric. This realization connects directly with the lower bound $\lambda_0(T_0)$ of T_0, and therefore the assumption of “finite energy integrals” can be dropped in applications to symmetric differential operators.

Since X is a Π_m space, in practice, one can obtain the maximal negative subspaces as Q_1-orthogonal complements of the maximal positive subspaces in X, which are often easier to describe. Now, making use of (2.31) and the expanded Phillips theory (see Theorem 2.3), we immediately obtain the following more explicit result which will be a basic for the case of differential operators in the next section.

Theorem 2.10. *Let T_0 be a symmetric operator with $\lambda_0(T_0) > 0$ and $\operatorname{def}(T_0) = m < \infty$, and Γ be a boundary mapping of T_0^* satisfying* (2.24). *Then an operator T is a maximal accretive extension of T_0 if and only if its adjoint, T^*, is a restriction of T_0^* to a domain of the form*

$$D(T^*) = \{u \in D(T_0^*) : \; 2(u_F, \phi_k)_D + \Gamma(u) B \alpha_k^* = 0, \; 1 \leqslant k \leqslant m\}, \tag{2.37}$$

where u_F is defined by (2.20) *and the elements $\phi_k \in H_F$ and row vectors $\alpha_k \in \mathbf{C}^{2m}$, $1 \leqslant k \leqslant m$, satisfy*

$$\hat{\phi}_k = \{\phi_k, \alpha_k\} \neq 0, \qquad 2(\phi_k, \phi_s)_D + \alpha_k B \alpha_s^* \begin{cases} = 0 & \text{if } k \neq s, \\ \leqslant 0 & \text{if } k = s. \end{cases} \tag{2.38}$$

Proof. As X is a Π_m space, all maximal positive subspaces of X have dimension m and are generated by a set $\{\hat{\phi}_1, \ldots, \hat{\phi}_m\}$, where all $\hat{\phi}_i$ in X satisfy (2.38). Each maximal negative subspace, $\hat{N}$, in X is the Q_1-orthogonal complement in X of such a subspace. By Theorems 2.3 and 2.9,

$$N = \{\{u, -T_0^* u\} \in G(-T_0^*) : \hat{u} = \{u_F, \Gamma(u)\} \in X \\ \text{satisfies } Q_1(\hat{u}, \hat{\phi}_i) = 0, 1 \leqslant i \leqslant m\}$$

is the graph of the adjoint of a maximal accretive extension of T_0, and all such extensions are found in this way. The proof is complete. $\square$

3. Maximal accretive differential operators

In this section we apply the expanded Phillips theory to the maximal accretive restriction problem for formally symmetric differential equations. In view of Theorem 2.10, the crux of the problem is to characterize the Friedrichs extension. Thus, we will appeal to the principal solutions of a Hamiltonian system, to which the associated differential equations can be transformed, and the work of Marletta and Zettl [17] for the Friedrichs extension of symmetric singular differential operators of order $2n$.

3.1. Preliminary theory

Let $I := [a, b)$, $-\infty < a < b \leqslant \infty$, be a half-open interval of the real line and let n be a positive integer. Assume that

$$1/p_0, p_1, \ldots, p_n, w \in L^1_{\text{loc}}(I, \mathbf{R}), \quad w > 0, \; p_0 > 0 \text{ a.e. on } I, \tag{3.1}$$

where $L^1_{\text{loc}}(I, \mathbf{R})$ denotes the set of real functions that are Lebesgue integrable on all compact subintervals of I. We define the quasi-derivatives $y^{[k]}$ as follows:

$$y^{[k]} = \begin{cases} y^{(k)}, & 0 \leqslant k \leqslant n-1, \\ p_0 y^{(n)}, & k = n, \\ p_{k-n} y^{(2n-k)} - \{y^{[k-1]}\}', & n+1 \leqslant k \leqslant 2n, \end{cases} \tag{3.2}$$

where $y^{(k)}$ is the usual kth derivative (cf. [10]). The symmetric quasi-differential expression we study here is given by

$$ly = w^{-1} y^{[2n]}. \tag{3.3}$$

For a more comprehensive discussion of quasi-differential equations the reader is referred to [10,19]. If the coefficient functions p_i, $0 \leqslant i \leqslant n$, are sufficiently smooth then l may also be written in the form

$$ly = \frac{1}{w(t)} \sum_{i=0}^{n} (-1)^i (p_{n-i}(t) y^{(i)})^{(i)}, \qquad t \in I, \tag{3.4}$$

which is more obviously symmetric. Note that conditions (3.1) ensure that the expression l is regular on $[a, c]$ for all c in (a, b). Furthermore, we assume that the endpoint b is singular, that is, at least one of the functions $w, 1/p_0, p_1, \ldots, p_n$ does not lie in $L^1([c, b))$ for some $c \in I$.

The differential expression l will be considered throughout the section in the weighted Hilbert space $L^2(w, I)$ of Lebesgue measurable functions which are square integrable with weight w and with inner product and norm defined by $(f, g) = \int_I f(t)\overline{g(t)}w(t)\,dt$ and $\|f\| = (f, f)^{1/2}$. Associated with the expression l, two differential operators $L_{\max}$, $L'_{\min}$, respectively called the *maximal operator*, *pre-minimal operator*, are defined as follows (see, e.g., [19, Section 17]): Let

$$D(L_{\max}) = \{y \in L^2(w, I) : y^{[k]} \in AC_{\text{loc}}(I), 0 \leqslant k \leqslant 2n-1, ly \in L^2(w, I)\},$$

$$D(L'_{\min}) = \{y \in D(L_{\max}) : y \text{ has compact support in } (a, b)\}.$$

Here $AC_{\text{loc}}(I)$ denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of I. Then

$$L_{\max} y = ly, \qquad y \in D(L_{\max}), \tag{3.5}$$

$$L'_{\min} y = ly, \qquad y \in D(L'_{\min}). \tag{3.6}$$

It is well known [19, Section 17] that both $D(L_{\max})$ and $D(L'_{\min})$ are dense in $L^2(w, I)$ (therefore, $L_{\max}$ has a unique adjoint $L^*_{\max}$), and $L'_{\min}$ has a closure $L_{\min}$ which is called the *minimal operator*. $L_{\min}$ is a closed, symmetric, densely defined operator in $L^2(w, I)$ and we have $L_{\min} = L^*_{\max}$ and $L_{\max} = L^*_{\min}$. Furthermore, from the basic conditions (3.1), it follows that the deficiency indices of $L_{\min}$ are $m_+ = m_- =: m$, say, where

$$n \leqslant m \leqslant 2n, \qquad m_\pm = \dim(\ker(L_{\max} \mp iI). \tag{3.7}$$

For proofs of these and other well-known facts reader is referred to [19,33].

Let

$$R_{2n}(y)(t) = (y(t), y^{[1]}(t), \ldots, y^{[2n-1]}(t)), \tag{3.8}$$

$$J_n = (\delta_{i,(n+1-j)})_{1 \leqslant i,j \leqslant n}, \qquad \hat{J}_{2n} = \begin{pmatrix} 0_n & J_n \\ -J_n & 0_n \end{pmatrix},$$

$$[y, z](t) = R_{2n}(y)(t)\hat{J}_{2n} R^*_{2n}(z)(t),$$

$$\langle y, z\rangle(t) = R_{2n}(y)(t) J_{2n} R^*_{2n}(z)(t), \tag{3.9}$$

where δ_{ij} denotes the Kronecker delta symbol. For any $y, z \in D(L_{\max})$ and $\alpha, \beta \in I$, the Green's formula ([19, p. 50]) and the Dirichlet formula [8] now respectively read as

$$\int_\alpha^\beta [(ly)\overline{z} - y(\overline{lz})]w\,dt = [y, z](\beta) - [y, z](\alpha), \tag{3.10}$$

$$\int_\alpha^\beta [(ly)\overline{z} + y(\overline{lz})]w\,dt = 2\sum_{i=0}^{n} \int_\alpha^\beta p_{n-i}\, y^{(i)}\overline{z}^{(i)}\,dt - \langle y, z\rangle(\beta) + \langle y, z\rangle(\alpha). \tag{3.11}$$

From [19, p. 71, 78], we have

$$D(L_{\min}) = \{y \in D(L_{\max}) : R_{2n}(y)(a) = 0, [y, z](b) = 0, \forall z \in D(L_{\max})\}. \tag{3.12}$$

3.2. The principal solutions and the Friedrichs extension

We consider Hamiltonian systems of the form

$$\tilde{J}_{2n}\mathbf{y}' = S(t, \lambda)\mathbf{y}, \qquad t \in [a, b), \tag{3.13}$$

where $\tilde{J}_{2n}$ is the $2n \times 2n$ symplectic matrix and S is a $2n \times 2n$ symmetric matrix given by

$$\tilde{J}_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \qquad S(t, \lambda) = \begin{pmatrix} S_{11}(t, \lambda) & S_{12} \\ S^T_{12} & S_{22}(t) \end{pmatrix}, \tag{3.14}$$

where S_{11} is real and symmetric for $\lambda \in \mathbf{R}$ and S_{22} is positive semidefinite. We partition the solution vector $\mathbf{y}$ as

$$\mathbf{y} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \qquad \mathbf{u}^T, \mathbf{v}^T \in \mathbf{C}^n.$$

Definition 3.1. The Hamiltonian system (3.13) is said to be *disconjugate* on an interval $(\alpha, \beta) \subseteq [a, b)$ if for *every* interval $(c, d) \subseteq (\alpha, \beta)$ whose endpoints are regular points of the differential equation, the boundary value problem

$$\tilde{J}_{2n}\mathbf{y}' = S(t, \lambda)\mathbf{y}, \qquad \mathbf{u}^T(c) = 0 \in \mathbf{C}^n, \quad \mathbf{u}^T(d) = 0 \in \mathbf{C}^n \tag{3.15}$$

has only the trivial solution.

Lemma 3.2. *Suppose that for some $\lambda \in \mathbf{R}$ and some $d \in [a, b)$ the Hamiltonian system* (3.13) *is disconjugate on an interval (d, b). Then for each $s \in (d, b)$ the matrix boundary value problem*

$$\tilde{J}_{2n} \begin{pmatrix} U \\ V \end{pmatrix}' = S(t, \lambda) \begin{pmatrix} U \\ V \end{pmatrix}, \qquad U(d) = I_n, \quad U(s) = 0_n, \tag{3.16}$$

(*in which U and V are $n \times n$ matrices*) *has a unique $2n \times n$ solution*

$$Y_s(t) =: \begin{pmatrix} U_s \\ V_s \end{pmatrix}. \quad \textit{Moreover } Y_b(t) := \lim_{s \to b} Y_s(t) =: \begin{pmatrix} U_b \\ V_b \end{pmatrix} \tag{3.17}$$

exists, uniformly for t in compact subsets of $[d, b)$, and is a solution of the Hamiltonian system.

Proof. See [27, Theorem 11.3, p. 331]. □

Remark. The solution Y_b defined by this process is called a *principal solution* of the Hamiltonian system.

For the differential equation $ly = \lambda y$, $\lambda \in \mathbf{R}$, an associated Hamiltonian system can be defined as follows. To each sufficiently smooth function y defined on $[a, b)$, we associate a vector $\mathbf{y} = \binom{\mathbf{u}}{\mathbf{v}}$ in which the components of $\mathbf{u}$ and $\mathbf{v}$ are given by

$$\mathbf{u} = (y^{[0]}, y^{[1]}, \ldots, y^{[n-1]})^T, \qquad \mathbf{v} = (y^{[n]}, y^{[n+1]}, \ldots, y^{[2n-1]})^T.$$

It may be shown that y satisfies the equation $ly = \lambda y$ if and only if $\mathbf{y}$ satisfies a Hamiltonian system of the type (3.13) (see [27, p. 343]).

Based on the above fact, it is now possible to obtain principal solutions for the equation $ly = \lambda y$. We compute a principal solution Y_b for the corresponding Hamiltonian system as a limit of solutions Y_s, $s \to b$. For each $1 \leqslant k \leqslant n$, the kth column of Y_b corresponds to the solution $y_s^{(k)}$ of the boundary value problem

$$ly = \lambda y, \qquad \mathbf{u}(d) = e_k, \quad \mathbf{u}(s) = 0,$$

where e_k^T is the kth standard basis vector of $\mathbf{C}^n$. Eq. (3.13) ensures that

$$y_k(t) := \lim_{s \to b} y_s^{(k)}(t) \tag{3.18}$$

exists, and is the solution of $ly = \lambda y$. We naturally call y_k the kth principal solution of $ly = \lambda y$.

Assumption 1. There exists $\lambda_0 > 0$ such that the differential equation $ly = \lambda_0 y$ is disconjugate on $[a, b)$.

Lemma 3.3. *Let Assumption* 1 *hold. Then*
(i) $\lambda_0(L_{\min}) > 0$;
(ii) *If* $\mu \leqslant \lambda_0$, *the equation* $ly = \mu y$ *is also disconjugate on* I *and its principal solutions all are in* $L^2(w, I)$.

Proof. These are given by [17, p. 413]. $\square$

Remark. By Lemma 3.3, Assumption 1 ensures that $\lambda_0(L_{\min}) > 0$, that is, the symmetric operator $L_{\min}$ is accretive. In fact, if the disconjugacy condition fails for every real λ then the minimal operator $L_{\min}$ will not be bounded below (see [27, p. 340]). In particular, when $n = 1$ and l is of limit-circle type, Assumption 1 is equivalent to the condition $\lambda_0(L_{\min}) > 0$ (see [21]).

Under Assumption 1 and $\operatorname{def}(L_{\min}) = m$, from [32, p. 115] and Lemma 3.3 we conclude that the equation $ly = 0$ possesses m linearly independent square integrable solutions with w on $[a, b)$, which contain n principal solutions. Therefore, we will denote the principal solutions by $\varphi_{m-n+1}, \ldots, \varphi_m$; denote the others by $\varphi_1, \ldots, \varphi_{m-n}$ which may be called the *nonprincipal solutions.* Note that if φ is a nonprincipal solution and ψ is a principal solution then $\varphi + c\psi$ is also a nonprincipal solution for any $c \in \mathbf{C}$.

Lemma 3.4. *If Assumption* 1 *holds, then the domain of the Friedrichs extension* L_F *of* $L_{\min}$ *is given by*

$$\begin{aligned} D(L_F) = \{y \in D(L_{\max}) : R_n(y)(a) = 0, \\ [y, \varphi_k](b) = 0, m - n + 1 \leqslant k \leqslant m\}. \end{aligned} \tag{3.19}$$

Proof. See [17, Theorem 12]. □

There seem to be $2n$ boundary conditions in (3.19). Indeed, because $\operatorname{def}(L_{\min}) = m$ $(n \leqslant m \leqslant 2n)$ and the Friedrichs extension L_F is self-adjoint, there exist m boundary conditions to describe the domain $D(L_F)$ of L_F. Furthermore, our aim is to use Theorem 2.10 to characterize the maximal accretive restrictions of $L_{\max}$, and therefore we have to find m boundary conditions for the Friedrichs extension.

Lemma 3.5. *Let Assumption* 1 *hold and let* $\varphi_1, \ldots, \varphi_{m-n}$ *be linearly independent nonprincipal solutions of the equation* $ly = 0$. *Then there exist* $m-n$ *principal solutions of* $ly = 0$, *denoted by* $\varphi_{m-n+1}, \ldots, \varphi_{2m-2n}$, *such that*

$$\operatorname{rank}\ [([\varphi_i, \varphi_j](b))_{1 \leqslant i,j \leqslant 2m-2n}] = 2m - 2n. \tag{3.20}$$

Proof. Let $y^{(j)}$, $1 \leqslant j \leqslant n$, and φ_i, $1 \leqslant i \leqslant m-n$, be linearly independent principal solutions and nonprincipal solutions of $ly = 0$ respectively. Then $y^{(1)}, \ldots, y^{(n)}, \varphi_1, \ldots, \varphi_{m-n}$ are linearly independent relative to $D(L_{\min})$. Let

$$\begin{aligned} \Delta &= \begin{pmatrix} ([\varphi_i, \varphi_j](b))_{1 \leqslant i,j \leqslant m-n} & ([\varphi_i, y^{(j)}](b))_{1 \leqslant i \leqslant m-n, 1 \leqslant j \leqslant n} \\ ([y^{(i)}, \varphi_j](b))_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m-n} & ([y^{(i)}, y^{(j)}](b))_{1 \leqslant i,j \leqslant n} \end{pmatrix} \\ &=: \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}. \end{aligned} \tag{3.21}$$

The proof similar to one in [31, Lemma 3] yields $\operatorname{rank} \Delta = 2m - 2n$. Furthermore, from [17, Theorem 10] each principal solution y in a neighborhood of $t = b$ has the following form:

$$y = g + f,$$

where the function f in $D(L_{\max})$ has compact support in $[a, b)$ and g is an element of $D(L_F)$. It follows from Lemma 3.4 that

$$[g, y^{(j)}](b) = 0 = [f, y^{(j)}](b), \qquad 1 \leqslant j \leqslant n. \tag{3.22}$$

This yields $\Delta_{22} = 0$. From the fact that $\Delta_{12} = -\Delta_{21}^*$ (see (3.9)), we see that $\operatorname{rank} \Delta_{12} = \operatorname{rank} \Delta_{21} = m-n$. Thus, there exist $m-n$ principal solutions, say $\varphi_{m-n+1}, \ldots, \varphi_{2m-2n}$, such that (3.20) holds, which finishes the proof. □

Keeping in mind (3.8) and (3.20), we define

$$R(y) = (R_{2n}(y)(a), r_{2m-2n}(y)(b)), \tag{3.23}$$

where

$$r_{2m-2n}(y)(b) = ([y, \varphi_1](b), \ldots, [y, \varphi_{2m-2n}](b)). \tag{3.24}$$

Here, $\varphi_1, \dots, \varphi_{m-n}$ and $\varphi_{m-n+1}, \dots, \varphi_{2m-2n}$ are respectively the nonprincipal solutions and principal solutions of $ly = 0$, which satisfy (3.20). By (3.12), Lemma 3.4 and Definition 2.6, it is easily verified that the linear mapping $R(\cdot) : D(L_{\max}) \mapsto \mathbf{C}^{2m}$ is the boundary mapping of $L_{\max}$.

The following theorem gives a new characterization of the Friedrichs extension of $L_{\min}$, which possesses m linearly independent boundary conditions.

Theorem 3.6. *Suppose that Assumption* 1 *holds and* φ_i, $m-n < i \leqslant 2m-2n$, *satisfying* (3.20) *are the principal solutions of* $ly = 0$. *Then the domain of the Friedrichs extension of* $L_{\min}$ *is given by*

$$D(L_F) = \{y \in D(L_{\max}) : R_n(y)(a) = 0,$$
$$[y, \varphi_k](b) = 0, m-n+1 \leqslant i \leqslant 2m-2n\}. \qquad (3.25)$$

Proof. Let D denote the right-hand of (3.25). We want to show that $D = D(L_F)$. It is easy to see from Lemma 3.4 that $D(L_F) \subseteq D$. To complete the proof we must show that D is the domain of a self-adjoint extension of $L_{\min}$.

Let χ_i, $i = 1, \dots, 2n$, be the functions in $D(L_{\max})$ which satisfy the conditions: $\chi_i^{[k-1]}(a) = \delta_{ik}$, $\chi_i(t) = 0$, for all $t \geqslant c$ with some $c \in (a, b)$. (For the existence of these functions, see [19, Section 17]). Clearly, $\chi_i \notin D(L_{\min})$. Let

$$\tilde{D} = D(L_{\min}) \dot{+} \operatorname{span}\{\chi_1, \dots, \chi_{2n}\}.$$

For any $y \in D(L_{\max})$ and $\tilde{z} \in \tilde{D}$ it is not difficult to see that $[y, \tilde{z}](b) = 0$. Furthermore, the proof of [31, Lemma 3] with an obvious modification shows that each y in $D(L_{\max})$ can be uniquely represented as

$$y = \tilde{y} + \sum_{i=1}^{2m-2n} c_i \varphi_i, \qquad \tilde{y} \in \tilde{D}. \qquad (3.26)$$

Note that φ_i, $1 \leqslant i \leqslant 2m-2n$, satisfy (3.20). Letting $\gamma = (c_1, \dots, c_{2m-2n})$ and $\Delta_{2m-2n} = ([\varphi_i, \varphi_j](b))_{1 \leqslant i,j \leqslant 2m-2n}$, we have

$$\gamma = r_{2m-2n}(y)(b) \Delta_{2m-2n}^{-1}. \qquad (3.27)$$

This together with (3.10), (3.20) and (3.26) implies that for any $y \in D(L_{\max})$,

$$\begin{aligned} 2i \operatorname{Im}(L_{\max} y, y) &= (L_{\max} y, y) - (y, L_{\max} y) \\ &= [y, y](b) - [y, y](a) \\ &= \gamma \Delta_{2m-2n} \gamma^* - R_{2n}(y)(a) \hat{J}_{2n} R_{2n}^*(y)(a) \end{aligned}$$

$$
\begin{aligned}
&= -r_{2m-2n}(y)(b)\Delta_{2m-2n}^{-1} r_{2m-2n}^{*}(y)(b) \\
&\quad - R_{2n}(y)(a)\hat{J}_{2n} R_{2n}^{*}(y)(a) \\
&= -R(y)\begin{pmatrix} \hat{J}_{2n} & 0_{2n\times(2m-2n)} \\ 0_{(2m-2n)\times 2n} & \Delta_{2m-2n}^{-1} \end{pmatrix} R^{*}(y) \\
&=: 2i R(y) A R^{*}(y).
\end{aligned} \tag{3.28}
$$

We note that $\varphi_{m-n+1}, \ldots, \varphi_{2m-2n}$ all are the principal solutions of the equation $ly = 0$ and therefore $[\varphi_i, \varphi_j](b) = 0,\ \ m-n+1 \leqslant i, j \leqslant 2m-2n$. Let

$$
M = \begin{pmatrix} I_n & 0_n & 0_{n\times(m-n)} & 0_{n\times(m-n)} \\ 0_{(m-n)\times n} & 0_{(m-n)\times n} & 0_{m-n} & I_{m-n} \end{pmatrix}. \tag{3.29}
$$

Then

$$
\operatorname{rank} M = m, \quad M A^{-1} M^{*} = 0, \quad D = \{ y \in D(L_{\max}) : M R^{*}(y) = 0 \}.
$$

Since $R(\cdot) : D(L_{\max}) \mapsto \mathbf{C}^{2m}$ is the boundary mapping of $L_{\max}$, applying [31, Lemma 4], we immediately obtain that D is the domain of a self-adjoint extension of $L_{\min}$. This completes the proof of Theorem 3.6. □

3.3. Maximal accretive differential operators

In the previous section we introduced the expanded Phillips theory and a concrete realization of the expanded space for a symmetric operator. In the last subsection we used the principal solutions of the equation $ly = 0$ to characterize the boundary conditions of the Friedrichs extension of $L_{\min}$, under Assumption 1. With these results, in this subsection we aim to characterize the maximal accretive extensions of the minimal operator $L_{\min}$.

Based on the Friedrichs extension L_F of $L_{\min}$ (see Theorem 3.6) and the boundary mapping $R(\cdot)$ of $L_{\max}$ (see (3.23)), from (2.31) we will establish the explicit realization of the expanded space X which associates the completing space H_F of $D(L_F)$ and the matrix B related to the boundary mapping $R(\cdot)$.

Lemma 3.7. *Let Assumption* 1 *hold and* $p_i \geqslant 0,\ \ 0 \leqslant i \leqslant n$. *Denote by* $W_0^{n,2}(I)$ *the completing space of* $D(L_F)$ *with respect to the inner product* $(L_F\cdot, \cdot)$. *Then*

$$
\begin{aligned}
W_0^{n,2}(I) = \Bigg\{ & y \in L^2(w, I) : y^{(n-1)} \in AC_{\mathrm{loc}}(I),\ \sum_{i=0}^{n} \int_I p_{n-i}(t) |y^{(i)}|^2 dt < \infty, \\
& R_n(y)(a) = 0,\ \lim_{t\to b} \left(\sum_{i=1}^{n} y^{(i-1)}(t) y_F^{[2n-i]}(t) \right) = 0, \forall\ y_F \in D(L_F) \Bigg\}
\end{aligned} \tag{3.30}
$$

and it is associated with the inner product

$$(y,z)_D = \sum_{i=0}^{n} \int_I p_{n-i}(t) y^{(i)} \bar{z}^{(i)}\, dt \qquad (y,z \in W_0^{n,2}(I)). \tag{3.31}$$

Proof. See [30, Lemma 2.6]. □

Lemma 3.8. *Let Assumption* 1 *hold. Then there are* $m-n$ *nonprincipal solutions, say* $\varphi_1, \ldots, \varphi_{m-n}$, *and n principal solutions, say* $\varphi_{m-n+1}, \ldots, \varphi_m$, *of the equation* $ly = 0$ *such that* $\varphi_1, \ldots, \varphi_{2m-2n}$ *satisfy* (3.20) *and*
(i) $\Phi_{11} := (R_n(\varphi_i)(a))_{1 \leqslant i \leqslant m-n} = 0$;
(ii) $\Phi_{21} := (R_n(\varphi_{m-n+i})(a))_{1 \leqslant i \leqslant n}$ *is a unitary matrix* (*i.e.*, $\Phi_{21}\Phi_{21}^* = I_n$).

Proof. We apply Lemma 3.5, letting $\psi_1, \ldots, \psi_{m-n}$ and $\psi_{m-n+1}, \ldots, \psi_m$ denote the nonprincipal solutions and principal solutions of $ly = 0$, respectively, in which $\psi_1, \ldots, \psi_{2m-2n}$ satisfy (3.20). We claim that rank $(R_n(\psi_i)(a))_{m-n < i \leqslant n} = n$. If not, there is a function $\psi := \sum_{i=1}^{n} c_i \psi_{m-n+i} \neq 0$ such that $R_n(\psi)(a) = 0$, which implies that 0 is an eigenvalue of L_F (see Lemma 3.4). This contradicts the inequality $\lambda_0(L_F) > 0$. So, the above claim is proved. Let

$$\begin{aligned}
\varphi'_{m-n+1} &= \psi_{m-n+1},\\
\varphi'_{m-n+2} &= \psi_{m-n+2} - \frac{[R_n(\psi_{m-n+2})(a),\, R_n(\varphi'_{m-n+1})(a)]}{[R_n(\varphi'_{m-n+1})(a),\, R_n(\varphi'_{m-n+1})(a)]}\, \varphi'_{m-n+1},\\
\cdots &= \cdots \qquad \cdots \qquad \cdots\\
\varphi'_m &= \psi_m - \frac{[R_n(\psi_m)(a),\, R_n(\varphi'_{m-n+1})(a)]}{[R_n(\varphi'_{m-n+1})(a),\, R_n(\varphi'_{m-n+1})(a)]}\, \varphi'_{m-n+1} - \cdots\\
&\quad - \frac{[R_n(\psi_m)(a),\, R_n(\varphi'_{m-1})(a)]}{[R_n(\varphi'_{m-1})(a),\, R_n(\varphi'_{m-1})(a)]}\, \varphi'_{m-1},
\end{aligned}$$

where $[\cdot,\cdot]$ is the usually inner product on $\mathbf{C}^n$. It is not hard to verify that $\{R_n(\varphi'_{m-n+i})(a)\}_{i=1}^{n}$ is a orthogonal set. Further, if we let $\varphi_i = \varphi'_i / ||R_n(\varphi'_i)(a)||$, then Φ_{21} is a unitary matrix. Moreover, for each ψ_i, $1 \leqslant i \leqslant m-n$, we can choose a principal solution $y_i := \sum_{k=1}^{n} c_k^{(i)} \varphi_{m-n+k}$ such that $R_n(\psi_i + y_i)(a) = 0$. Clearly, $\varphi_i := \psi_i + y_i$ is a nonprincipal solution of $ly = 0$ and $\Phi_{11} = 0$, thus completing the proof. □

Let

$$\begin{aligned}
\Phi_{12} &= (\varphi_i^{[n+j-1]}(a))_{1 \leqslant i \leqslant m-n, 1 \leqslant j \leqslant n}, \quad & \Phi_{22} &= (\varphi_{m-n+i}^{[n+j-1]}(a))_{1 \leqslant i, j \leqslant n},\\
\Phi_{32} &= (\varphi_{m-n+i}^{[n+j-1]}(a))_{1 \leqslant i \leqslant m-n, 1 \leqslant j \leqslant n}, \quad & \Phi_{31} &= (R_n(\varphi_i)(a))_{m-n < i \leqslant 2m-2n},
\end{aligned}$$

$$B = \begin{pmatrix} 0_n & J_n & 0_{n\times(m-n)} & 0_{n\times(m-n)} \\ J_n & J_n\Phi_{21}^*\Phi_{22} + \Phi_{22}^*\Phi_{21}J_n & 2\Phi_{12}^* & 0_{n\times(m-n)} \\ 0_{(m-n)\times n} & 2\Phi_{12} & 0_{m-n} & -\Phi_{12}J_n\Phi_{31}^* \\ 0_{(m-n)\times n} & 0_{(m-n)\times n} & -\Phi_{31}J_n\Phi_{12}^* & 0_{m-n} \end{pmatrix}. \tag{3.32}$$

Lemma 3.9. *Let Assumption* 1 *hold. For any* $y \in D(L_{\max})$ *we have*

$$2\mathrm{Re}(L_{\max}y, y) = 2(y, y)_D^F + R(y)B^{-1}R^*(y), \tag{3.33}$$

where $(y, y)_D^F$ *and* $R(y)$ *are defined by* (2.21) *and* (3.23), *respectively, and* B^{-1} *is the inverse matrix of* B.

Proof. From Lemma 2.5, each y in $D(L_{\max})$ can be uniquely represented as

$$y = y_F + \sum_{i=1}^{m} c_i\varphi_i, \qquad y_F \in D(L_F). \tag{3.34}$$

By (3.9), (3.10) and (3.25), for each $y \in D(L_{\max})$, we have

$$\begin{aligned}(L_{\max}y, y) &= \left(L_F y_F, y_F + \sum_{i=1}^{m} c_i\varphi_i\right)\\ &= (y, y)_D^F + \left[y_F, \sum_{i=1}^{m} c_i\varphi_i\right](b) - \left[y_F, \sum_{i=1}^{m} c_i\varphi_i\right](a)\\ &= (y, y)_D^F + \sum_{i=1}^{m-n} \bar{c}_i[y_F, \varphi_i](b) - R_{2n}(y_F)(a)\hat{J}_{2n}R_{2n}^*\left(\sum_{i=1}^{m} c_i\varphi_i\right)(a).\end{aligned}$$

Letting $R_0(y) = (y_F^{[n]}(a), \ldots, y_F^{[2n-1]}(a), r_{m-n}(y_F)(b), c_1, \ldots, c_m)$, we get

$$2\mathrm{Re}(L_{\max}y, y) = 2(y, y)_D^F + R_0(y)B_0R_0^*(y), \tag{3.35}$$

where

$$B_0 = \begin{pmatrix} 0_m & B_{00} \\ B_{00}^* & 0_m \end{pmatrix} \quad \text{with } B_{00} = \begin{pmatrix} 0_{n\times(m-n)} & J_n\Phi_{21}^* \\ I_{m-n} & 0_{(m-n)\times n} \end{pmatrix}. \tag{3.36}$$

Note that by Green's formula, $[\varphi_i, \varphi_j](a) = [\varphi_i, \varphi_j](b),\ 1 \leqslant i, j \leqslant m$. It follows from (3.25) and (3.34) that

$$R_{2n}(y)(a) = (0_{1\times n}, y_F^{[n]}(a), \ldots, y_F^{[2n-1]}(a)) + (c_1, \ldots, c_m) \begin{pmatrix} 0_{(m-n)\times n} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix},$$

$$r_{2m-2n}(y)(b) = ([y_F, \varphi_1](b), \ldots, [y_F, \varphi_{m-n}](b), 0_{1\times(m-n)}) + (c_1, \ldots, c_m) \begin{pmatrix} 0_{(m-n)\times n} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \hat{J}_{2n} \begin{pmatrix} 0_{n\times(m-n)} & \Phi_{31}^* \\ \Phi_{12}^* & \Phi_{32}^* \end{pmatrix}.$$

Thus $R(y) = R_0(y)F$ with

$$F = \begin{pmatrix} 0_n & I_n & 0_{n\times(m-n)} & 0_{n\times(m-n)} \\ 0_{(m-n)\times n} & 0_{(m-n)\times n} & I_{m-n} & 0_{m-n} \\ 0_{(m-n)\times n} & \Phi_{12} & 0_{m-n} & -\Phi_{12} J_n \Phi_{31}^* \\ \Phi_{21} & \Phi_{22} & \Phi_{21} J_n \Phi_{12}^* & 0_{n\times(m-n)} \end{pmatrix}. \tag{3.37}$$

Substituting this into (3.35), by a simple calculation we obtain (3.33) and complete the proof. □

Now, by Theorem 2.10 and Lemmas 3.7, 3.9, we can directly construct the expanded space X associated with $L_{\max}$, which is defined by

$$X = W_0^{n,2}(I) \times \mathbf{C}^{2m} := (W_0^{n,2}(I) \times \mathbf{C}^{2m},\ Q_1(\cdot, \cdot)) \tag{3.38}$$

with the indefinite inner product

$$Q_1(\hat{u}_1, \hat{u}_2) = -2(u_1, u_2)_D - \alpha_1 B^{-1} \alpha_2^*, \tag{3.39}$$

where $\hat{u}_i = \{u_i, \alpha_i\} \in X,\ u_i \in W_0^{n,2}(I),\ \alpha_i \in \mathbf{C}^{2m},\ i = 1, 2$.

Applying the expanded Phillips theory (see Theorems 2.3 and 2.10), we are ready to state one of our main results of this section.

Theorem 3.10. *Let* $\operatorname{def}(L_{\min}) = m$, *Assumption* 1 *hold and* $p_i \geqslant 0,\ 0 \leqslant i \leqslant n$. *Then an operator L is a maximal accretive restriction of* $L_{\max}$ *if and only if L is a restriction of* $L_{\max}$ *to a domain of the form*

$$D(L) = \{y \in D(L_{\max}) : 2(y_F, \psi_k)_D + R(y)B^{-1}\alpha_k^* = 0,\ 1 \leqslant k \leqslant m\}, \tag{3.40}$$

where y_F *and* $R(y)$ *are defined by* (2.20) *and* (3.23), *respectively,* B^{-1} *is the inverse matrix of B defined by* (3.32), *and the functions* $\psi_k \in W_0^{n,2}(I)$ *and row vectors* $\alpha_k \in \mathbf{C}^{2m},\ 1 \leqslant k \leqslant m$, *satisfy*

$$\hat{\psi}_k := \{\psi_k, \alpha_k\} \neq 0, \qquad 2(\psi_k, \psi_s)_D + \alpha_k B^{-1} \alpha_s^* \begin{cases} = 0 & \text{if } k \neq s, \\ \leqslant 0 & \text{if } k = s. \end{cases} \tag{3.41}$$

Theorem 3.10 characterizes all the maximal accretive restrictions L of $L_{\max}$. The significance of the construction of such restrictions L that all maximal accretive extensions of $L_{\min}$ then can be identified in L^* (see [6, p. 119]). We are in a position to describe the maximal accretive extensions of $L_{\min}$ explicitly, via the method of Brown and Krall [4]. For a detailed discussion of this method concerning the maximal accretive extensions of differential operators the reader is referred to [8, Appendix].

For a given $\gamma \in \mathbf{C}^m$ and $\Psi = (\psi_1, \ldots, \psi_m)^T$, $\psi_i \in W_0^{n,2}(I)$, the partial adjoint expressions for l in this case are defined by

$$l_k^+ z_\gamma = \begin{cases} z_\gamma^{[k]}, & 0 \leqslant k \leqslant n-1, \\ (z_\gamma - \gamma\Psi)^{[k]}, & n \leqslant k \leqslant 2n. \end{cases} \tag{3.42}$$

Further, let

$$D(L_{\max}^+) = \Big\{ z_\gamma \in L^2(w, I) : \text{there exists } \gamma \in \mathbf{C}^m \text{ such that}$$
$$l_k^+ z_\gamma \in AC_{\mathrm{loc}}(I), 0 \leqslant k \leqslant 2n-1, (1/w) l_{2n}^+ z_\gamma \in L^2(w, I) \Big\}.$$

It should be noted that if $z_\gamma \in D(L_{\max}^+)$ then $z_\gamma - \gamma\Psi \in D(L_{\max})$ (see (3.30) and (3.5)). Let $\varphi_i, 1 \leqslant i \leqslant 2m-2n$, be the solutions of $ly = 0$ satisfying (3.20). Set $\Phi_{2m-2n} = ([\varphi_i, \varphi_j](b))_{1 \leqslant i,j \leqslant 2m-2n}$,

$$R_+(z_\gamma) = (R_{2n}^+(z_\gamma)(a), r_{2m-2n}^+(z_\gamma)(b)), \quad A = \begin{pmatrix} \hat{J}_{2n} & 0 \\ 0 & \Phi_{2m-2n}^{-1} \end{pmatrix}, \tag{3.43}$$

where $\hat{J}_{2n}$ has been defined by (3.9), $z_\gamma \in D(L_{\max}^+)$ and

$$R_{2n}^+(z_\gamma)(a) = ((l_0^+ z_\gamma)(a), \ldots, (l_{2n-1}^+ z_\gamma)(a)),$$
$$r_{2m-2n}^+(z_\gamma)(b) = ([z_\gamma - \gamma\Psi, \varphi_1](b), \ldots, [z_\gamma - \gamma\Psi, \varphi_{2m-2n}](b)).$$

Note that $R_+(z_\gamma) = R(z_\gamma - \gamma\Psi)$, and for any $y \in D(L_{\max})$, $z_\gamma \in D(L_{\max}^+)$, by (3.10), (3.28) and (3.30) we have

$$\int_a^b [(ly)\overline{z_\gamma} - y(1/w)(\overline{l_{2n}^+ z_\gamma})] w \, dt = -R(y) A R_+^*(z_\gamma) + (y_F, \gamma\Psi)_D. \tag{3.44}$$

With the above notations, by [8, Theorem A.8], suitably modified, we have

Theorem 3.11. *Let the assumptions be as in Theorem* 3.10. *Then an operator L is a maximal accretive extension of $L_{\min}$ if and only if it has the form*

$$D(L)=\left\{z_\gamma \in D(L^+_{\max}) : R_+(z_\gamma) = \xi K B^{-1} A^{-1}\right\},$$

$$Lz_\gamma=\frac{1}{w} l^+_{2n} z_\gamma, \quad z_\gamma \in D(L), \tag{3.45}$$

where ξ in $\mathbf{C}^m$ is any vector equivalent to γ in the sense that $\xi\Psi = (1/2)\gamma\Psi$. Here $R_+(z_\gamma)$, A and B are defined by (3.43) *and* (3.32) *respectively, $K = (\alpha_i)_{1\leqslant i\leqslant m}$, and the functions $\psi_k \in W_0^{n,2}(I)$ and row vectors $\alpha_k \in \mathbf{C}^{2m}$, $1\leqslant k\leqslant m$,* satisfy (3.41).

3.4. Two examples

Firstly, we consider the special case of a second-order Sturm–Liouville differential expression on $[a, b)$:

$$ly = \frac{1}{w}[-(p_0 y')' + p_1 y]. \tag{3.46}$$

Here we assume that the coefficients w, p_0 and p_1 satisfy the basic conditions (3.1) for $n = 1$. Further, assume that l is singular at the endpoint b and is of limit-circle type, that is, $\operatorname{def}(L_{\min}) = 2$. In this case Assumption 1 is equivalent to the condition $\lambda_0(L_{\min}) > 0$ (see [21, p. 546]). Thus, if $\lambda_0(L_{\min}) > 0$ and $\operatorname{def}(L_{\min}) = 2$, by Lemma 3.8, there exist a real nonprincipal solution φ_1 and a real principal solution φ_2 of the equation $ly = 0$ such that

$$R_2(\varphi_1)(a) = (0, 1), \quad \varphi_2(a) = -1, \quad [\varphi_1, \varphi_2](b) = 1. \tag{3.47}$$

Let

$$R(y) = (y(a), y^{[1]}(a), [y, \varphi_1](b), [y, \varphi_2](b)),$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -2\varphi_2^{[1]}(a) & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.48}$$

It is easy to verify from (2.20) and (3.43) that $y = y_F + c_1\varphi_1 + c_2\varphi_2$ with $c_1 = [y, \varphi_2](b)$ and $c_2 = -y(a)$. By Lemma 3.9, for any $y \in D(L_{\max})$,

$$2\mathrm{Re}(L_{\max} y, y) = 2(y, y)^F_D + R(y)B^{-1}R^*(y). \tag{3.49}$$

Lemma 3.12. *Let* $\lambda_0(L_{\min}) > 0$, *the endpoint* b *be of limit-circle type and* $h = \varphi_1 + \varphi_2$. *Denote by* $W_0^{1,2}(I)$ *the completing space of* $D(L_F)$ *with respect to the inner product* $(L_F \cdot, \cdot)$. *Then*

$$W_0^{1,2}(I) = \left\{ y \in AC_{\mathrm{loc}}(I) : y(a) = 0 = \lim_{t \to b} \frac{y}{h}(t) \text{ exists}, \ \sqrt{p_0}\, h(\frac{y}{h})' \in L^2(I) \right\}$$

and it is associated with the inner product

$$(y_1, y_2)_D := \int_I p_0 h^2 (\frac{y_1}{h})'(\frac{\bar{y}_2}{h})'\, dt \qquad (y_1, y_2 \in W_0^{1,2}(I)). \tag{3.50}$$

Proof. See [30, Lemma 2.7]. □

Thus, from Lemma 3.12 and Theorem 3.10, we have

Corollary 3.13. *Let* $\lambda_0(L_{\min}) > 0$ *and* $\operatorname{def}(L_{\min}) = 2$. *An operator* L *is a maximal accretive restriction of* $L_{\max}$ *if and only if it is a restriction of* $L_{\max}$ *to a domain of the form*

$$D(L) = \{ y \in D(L_{\max}) : 2(y_F, \psi_k)_D + R(y) B^{-1} \alpha_k^* = 0, k = 1, 2 \}, \tag{3.51}$$

where $y_F = y - [y, \varphi_2](b)\varphi_1 + y(a)\varphi_2$, $R(y)$ *and* B *are defined as* (3.48) *and* $\psi_k \in W_0^{1,2}(I)$, $\alpha_k \in \mathbf{C}^4$, $1 \leqslant k \leqslant 2$, *satisfy*

$$\hat{\psi}_k = \{\psi_k, \alpha_k\} \neq 0 \quad \text{and} \quad 2(\psi_k, \psi_s)_D + \alpha_k B^{-1} \alpha_s^* \begin{cases} = 0 & \text{if } k \neq s, \\ \leqslant 0 & \text{if } k = s. \end{cases} \tag{3.52}$$

This corollary gives the characterization of all maximal accretive restrictions of the Sturm–Liouville operator $L_{\max}$ in the limit-circle type. This combined with Theorem 3.11 solves the open problem of Evans and Knowles in [8, p. 265].

Secondly, we consider the case of square of a Surm–Liouville expression. Let τ be the differential expression defined by $\tau y = -y'' + qy$ on $I := [a, b)$, where the real function $q \in C^2(I)$. Assume that the endpoint b is singular and τ is of limit-circle type at b. We consider the formal square τ^2 of τ, defined by

$$ly := \tau^2 y = \tau(\tau y) = y^{(4)} - (2qy')' + (q^2 - q'')y \qquad \text{on } I. \tag{3.53}$$

The differential expression l is a special type of (3.4) in the case $n = 2$. Let $L_{\min}$ and $L_{\max}$ be the minimal and maximal operators of l in $L^2(I)$, respectively. It is well known [9, p. 180] that the differential expression l is singular at b, and def $(L_{\min}) = 4$

(i.e. $m=4$, see (3.7)). Similar to the proof of [29, Lemma 2.1] we conclude that

$$\lambda_0(L_{\min})>0, \qquad L_F=T_{\max}(\tau)T_{\min}(\tau), \tag{3.54}$$

where L_F is the Friedrichs extension of $L_{\min}$, and $T_{\min}(\tau)$ and $T_{\max}(\tau)$ are the minimal and maximal operators associated with the expression τ. By Definition 3.1 and (3.53), we easily see that, for some positive number λ_0, the equation $ly=\lambda_0 y$ is disconjugate on I, that is, Assumption 1 holds. Thus, the principal solutions of $ly=0$ exist (see Lemma 3.2).

In order to describe the maximal accretive differential operators of l, by Theorem 3.10, we aim to find the matrix B (see (3.32)) and the space $W_0^{2,2}(I)$ (see (2.19)) corresponding to the differential expression l.

Suppose that u_1,u_2 are the real solutions of the equation $\tau y=0$ such that

$$R_2(u_1)(a)=(1,0), \qquad R_2(u_2)(a)=(0,1). \tag{3.55}$$

Let $k=\frac{1}{2}\int_a^b u_1u_2\,dt$ and suppose that u_3,u_4 are the real solutions of the inhomogeneous differential equations $\tau y=u_2, \tau y=u_1$, respectively, which satisfy the following initial conditions

$$R_2(u_3)(a)=(0,k), \qquad R_2(u_4)(a)=(-k,0). \tag{3.56}$$

It is easy to check that u_i in $L^2(I), 1\leqslant i\leqslant 4$, are the linearly independent solutions of $ly=0$.

In what follows, let $[y,z]_2(t)=R_2(y)(t)\hat{J}_2R_2^*(z)(t)$ for any $y,z\in D(T_{\max}(\tau))$ and $t\in I$.

Lemma 3.14. *Let*

$$\begin{aligned}
&\varphi_1=\begin{cases} u_3 & \textit{if } k=0,\\ \frac{1}{k}u_4+u_1 & \textit{if } k\neq 0,\end{cases} \qquad \varphi_2=\begin{cases} u_4 & \textit{if } k=0,\\ \frac{1}{k}u_3-u_2 & \textit{if } k\neq 0,\end{cases}\\
&\varphi_3=\frac{1}{(u_2,u_2)}\left(u_3+[u_3,u_2]_2(b)u_1-[u_3,u_1]_2(b)u_2\right),\\
&\varphi_4=\frac{-1}{(u_1,u_1)}\left(u_4+[u_4,u_2]_2(b)u_1-[u_4,u_1]_2(b)u_2\right).
\end{aligned} \tag{3.57}$$

Then φ_3,φ_4 are the principal solutions of $ly=0$ such that $\Phi_{21}=I_2$ (see Lemma 3.8 (ii)) *and φ_1,φ_2 are the nonprincipal solutions such that $\Phi_{11}=0$ (see Lemma* 3.8 (i)).

Proof. Let $s\in(a,b)$, $k(s)=\frac{1}{2}\int_a^s u_1u_2\,dt$, and suppose that $u_{3,s},u_{4,s}$ are the real solutions of the equations $\tau y=u_2, \tau y=u_1$, respectively, which satisfy the initial conditions $R_2(u_{3,s})(a) = (0, k(s)),\ \ R_2(u_{4,s})(a) = (-k(s), 0)$. Then

$$\varphi_{3,s} = \frac{1}{\int_a^s u_2^2\,dt}\left(u_{3,s} + [u_{3,s}, u_2]_2(s)u_1 - [u_{3,s}, u_1]_2(s)u_2\right),$$

$$\varphi_{4,s} = \frac{-1}{\int_a^s u_1^2\,dt}\left(u_{4,s} + [u_{4,s}, u_2]_2(s)u_1 - [u_{4,s}, u_1]_2(s)u_2\right),$$

are the solutions of the boundary value problem

$$ly = 0, \quad R_2(y)(a) = e_k, \quad R_2(y)(s) = 0,$$

where $e_k,\ k = 1, 2$, is kth standard basis vector of $\mathbf{C}^2$, which implies

$$\varphi_3 = \lim_{s \to b^-} \varphi_{3,s}, \quad \varphi_4 = \lim_{s \to b^-} \varphi_{4,s}.$$

It follows from (3.18) that φ_3, φ_4 are the principal solutions of $ly = 0$ such that $\Phi_{21} = I_2$ (see Lemma 3.8(ii)). Clearly, $\varphi_i, 1 \leqslant i \leqslant 4$, are the linearly independent solutions of $ly = 0$ and therefore φ_1, φ_2 are the nonprincipal solutions such that $\Phi_{11} = 0$ (see Lemma 3.8(i)), thus completing the proof. $\square$

Combined with (3.32) and $m = 4$, Lemma 3.14 ensures that

$$\Phi_{31} = \Phi_{21} = I_2,\ \ \Phi_{32} = \Phi_{22} = (\varphi_{2+i}^{[1+j]}(a))_{1 \leqslant i,j \leqslant 2},\ \ \Phi_{12} = (\varphi_i^{[1+j]}(a))_{1 \leqslant i,j \leqslant 2},$$

$$B = \begin{pmatrix} 0_2 & J_2 & 0_2 & 0_2 \\ J_2 & J_2\Phi_{22} + \Phi_{22}^* J_2 & 2\Phi_{12}^* & 0_2 \\ 0_2 & 2\Phi_{12} & 0_2 & -\Phi_{12}J_2 \\ 0_2 & 0_2 & -J_2\Phi_{12}^* & 0_2 \end{pmatrix}. \tag{3.58}$$

Lemma 3.15. *Denote by $W_0^{2,2}(I)$ the completing space of $D(L_F)$ with respect to the inner product $(L_F\cdot, \cdot)$. Then*

$$W_0^{2,2}(I) = \{y \in L^2(I) : y' \in AC_{\mathrm{loc}}(I),\ \ y'' - qy \in L^2(I),$$
$$R_2(y)(a) = 0, [y, z]_2(b) = 0 \ \ for\ all\ \ z \in D(T_{\max}(\tau))\} \tag{3.59}$$

and it is associated with the inner product

$$(y_1, y_2)_D := \int_I (-y_1'' + qy_1)(-\bar{y}_2'' + q\bar{y}_2)\,dt \quad (y_1, y_2 \in W_0^{2,2}(I)). \tag{3.60}$$

Proof. Since $D(L_F) \subset D(T_{\min}(\tau))$, it follows from Green formula (cf. (3.5)) that for any $y \in D(L_F)$,

$$\begin{aligned}\lambda_0(L_{\min})||y||^2 \leqslant (L_F y, y) &= (T_{\max}(\tau)T_{\min}(\tau)y, y)\\ &= (T_{\min}(\tau)y, T_{\min}(\tau)y)\\ &=: (y, y)_D,\end{aligned} \tag{3.61}$$

which yields that $(D(T_{\min}(\tau)), (\cdot,\cdot)_D)$ is a Hilbert space since $T_{\min}(\tau)$ is a close symmetric operator. Notice that $T_{\min}(\tau)$ is the closure of the realization of τ to $C_0^\infty(a,b)$ (see [19, Section 17]). Then $C_0^\infty(a,b) \subset D(L_F) \subset D(T_{\min}(\tau))$ and therefore $W_0^{2,2}(I)$ with respect with the inner product (3.60) is the completing space of $D(L_F)$, thus completing the proof. □

With the above results, by Theorem 3.10, we have immediately the following corollary.

Corollary 3.16. *Let* $q \in C^2(I)$ *and* $\operatorname{def}(L_{\min}) = 4$. *Then an operator* L *is a maximal accretive restriction of* $L_{\max}$ *if and only if it is a restriction of* $L_{\max}$ *to a domain of the form*

$$D(L) = \{y \in D(L_{\max}) : 2(y_F, \psi_k)_D + R(y)B^{-1}\alpha_k^* = 0, 1 \leqslant k \leqslant 4\}, \tag{3.62}$$

where y_F, $R(y)$ *and* B *are defined as* (2.10), (3.23) *and* (3.58) *respectively, and* $\psi_k \in W_0^{2,2}(I)$, $\alpha_k \in \mathbf{C}^8$, $1 \leqslant k \leqslant 4$, *satisfy*

$$\hat{\psi}_k = \{\psi_k, \alpha_k\} \neq 0 \quad \mathit{and} \quad 2(\psi_k, \psi_s)_D + \alpha_k B^{-1}\alpha_s^* \begin{cases} = 0 & \mathit{if}\ k \neq s, \\ \leqslant 0 & \mathit{if}\ k = s. \end{cases} \tag{3.63}$$

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